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Critical Remarks and an Open Question in the Present Theory of the Electromagnetic Self-Force

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It is asked whether the presently accepted theory of the classical EM self-force, the Lorentz-Dirac-Rohrlich theory, is "effectively analytic" in the classical radius r_c, that is, when solutions are modified by the suppression of terms $\propto e^{\tau/r_0}$, $r_0 \equiv \frac{2}{3}r_c$, which leads to practically unobservable noncausal behavior in proper time intervals $\Delta\tau \approx 10^{-23}$ sec, whether they are analytic in r_0 at $r_0 = 0$ and do agree with perturbation theory. Perturbation theory—assumed power series in r_0 – is known to give a good account of experimentally observed motion. After this open question is made precise, a tentative negative answer is given. A real disagreement with perturbation theory would cast considerable doubt on a theory of the EM self-force.

1. *Physical Remarks*

The present theory (Rohrlich, 1965) of the motion of a classical charged particle of charge q and total mass m under an external electromagnetic field $F_{\text{ext}}{}^{\mu}{}_{\nu}(x)$ and its self-field is the Lorentz-Dirac (LD) equation[1]

$$\ddot{x}^{\mu} = r_0\Gamma^{\mu} + (q/m)F_{\text{ext}}{}^{\mu}{}_{\nu}(x)\dot{x}^{\nu}, \qquad \Gamma^{\mu} \equiv \dddot{x}^{\mu} - \ddot{x}^2\dot{x}^{\mu} \tag{1.1}$$

where the dot means the derivative with respect to the proper time, τ, and $r_0 \equiv \frac{2}{3}q^2/4\pi m$, together with the initial data

$$x^{\mu}(\tau_0) = x_0{}^{\mu}, \qquad \dot{x}^{\mu}(\tau_0) = v_0{}^{\mu} \tag{1.2a}$$

and the Rohrlich asymptotic boundary condition

$$a^{\mu}(\tau)\underset{\tau\to\pm\infty}{\to} 0, \tag{1.2b}$$

where $v^{\mu} \equiv \dot{x}^{\mu}$, $a^{\mu} \equiv \ddot{x}^{\mu}$.

[1] We use the metric $(+++-)1$ with $x^4 \equiv t$. Units: $c = 1$. Often indices will be suppressed, e.g., v, $\ddot{x}$ stand for v^{μ}, $\ddot{x}^{\mu}$ etc. When components are written out, the order is $v = (v^1, v^2, v^3, v^4)$.

One may have reservations about the derivation[2] of the LD equation; however, here we wish merely, given the theory (1.1) *cum* (1.2), to examine its consequences.

Rewrite (1.1) as

$$\dddot{x}^\mu = \ddot{x}^2\dot{x}^\mu + \ddot{x}^\mu/r_0 - (1/r_0)\, A^\mu{}_\nu(x)\dot{x}^\nu \equiv f^\mu(x, v, a)$$
$$A^\mu{}_\nu \equiv (q/m)\, F_{\text{ext}}{}^\mu{}_\nu \tag{1.3}$$

This is an ordinary (nonlinear) third-order differential equation. Theorems exist (Coddington and Levinson, 1955) to guarantee a unique solution with any intial data $\{x_0, v_0, a_0\}$ under unrestrictive and physically easily realizable conditions on the external field. Namely, on the 12-dimensional space of points $X^a \equiv (x^1, \ldots, x^4, v^1, \ldots, v^4, a^1, \ldots, a^4)$, that a certain vector function $F^a(X) \equiv (v^1, \ldots, v^4, a^1, \ldots, a^4, f^1, \ldots, f^4)$ be continuous and Lipschitz in a domain of X space with respect to the norm $|X| \equiv \Sigma_a |X^a|$. Here $f^\mu(X)$ is defined in (1.3).

From now on we assume that all F_{ext} considered are such that these conditions are satisfied. Instead of initial data $\{x_0, v_0, a_0\}$, initial data $\{x_0, v_0\}$ plus the asymptotic condition $a(\tau) \to 0$ as $\tau \to \pm\infty$ are actually imposed in the LDR theory (1.1), (1.2) to prohibit runaway behavior at infinity; however, in the applications we have in mind, the boundary conditions (1.2) can be replaced by equivalent initial data,[3] so that the known existence and uniqueness theorems apply.

The physical troubles with the LDR theory seem all to stem from the fact that (1.3) is a third-order equation. The smallness of the parameter r_0 ($\approx 10^{-13}$ cm for an electron) helps here by making it "almost" a second-order equation, *cf.* (1.1). There are two main types: (a) runaway behavior for some times, violations of causality in the form of preacceleration and predeceleration; (b) Solutions in general not analytic in r_0 at $r_0 = 0$, hence no obvious connection of solutions with perturbation theory. These will be made more precise hereafter.

Let us comment first physically on (a). In the *free case* $A^\mu{}_\nu \equiv 0$, any solution $\dot{x}^\mu$ can be put into form[4]

$$\dot{x}^\mu = (0, 0, \sinh(Ae^{\tau/r_0} + B), \cosh(Ae^{\tau/r_0} + B)) \tag{1.4}$$

[2] Space forbids more than just mention here. (1) The shrinking spheres used in the limit procedure are centered at the particle at the field time rather than at the retarded time. (2) The inevitable arbitrary division of radiated 4-momentum into that lost by the particle and that lost by the field inside a sphere. (3) Conservation of a total (field + particle) 4-momentum belonging to an amputated, possibly wrong Lagrangian.

[3] Remember that "initial" data need not all be at the same proper time. Also the zero of τ is arbitrary by translation-invariance in τ. Hereafter we take $x^4(0) = 0$ and *then* impose initial data.

[4] Rohrlich (1965), Section 6–10.

where A and B are arbitrary constants, by a Lorentz transformation. These are accelerating [at a prodigious rate (!), hence called *runaway*] if and only if $A \neq 0$. We shall need the following theorem:

> *Theorem.* A free solution is uniform motion $[a^\mu(\tau) \equiv 0]$ iff $a^\mu = 0$ at some τ, or runaway iff $a^\mu \neq 0$ at some τ.

Proof. We can derive $a^2 = (A/r_0)^2 e^{2\tau/r_0}$ from (1.4). Hence $A = 0(\neq 0)$ iff a^2 is ever zero (nonzero). However, a^μ is spacelike, so $a^2 = 0 \Leftrightarrow a^\mu = 0$, Q.E.D.

From now on we restrict ourselves to external fields which are turned on only for a finite time interval. Precisely, let Supp $A^\mu{}_\nu(x) \subset$ the open strip $0 < t < t_1$. Now consider a solution of (1.1) *cum* (1.2). There is an initial free region I: $t < 0$, a driven region II: $0 < t < t_1$, and a final free region III: $t > t_1$. The asymptotic condition (1.2b) is *equivalent* by the Theorem to the initial data $a_0 = 0$ if we take our initial point $\tau = \tau_0$ in free region III. Now let us watch the solution as time and proper time flow backwards from τ_0, see Figure 1. The motion is uniform in region III by the Theorem. However, when the trajectory enters region II (at $\tau = \tau_1$, say) nonuniform motion in general begins, so that at the point of leaving region II (at $\tau = 0$, say), $a(0) \neq 0$ in general. But the subsequent free motion in region I is determined uniquely by the initial data $\{x(0), v(0), a(0) \neq 0\}$. Hence it is runaway by the Theorem. That is, there is preacceleration in free region I, and this of course can be made to last as long as desired. Similarly, predeceleration sets in around $\tau \lesssim \tau_1$. See Figure 2.

Actual solutions (see later) show that these violations of causality are *effectively* confined to proper time intervals of order $r_0 \approx 10^{-23}$ sec and

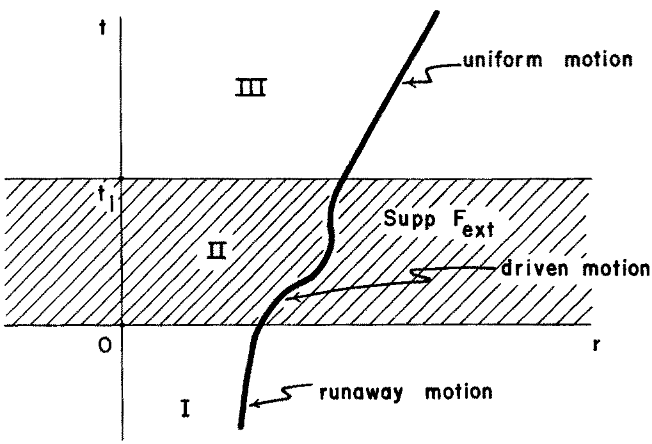

Figure 1—Typical trajectory with finite-duration external field.

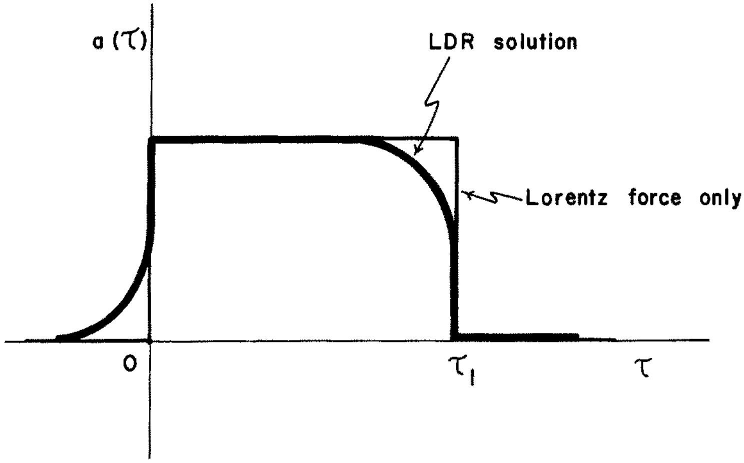

Figure 2—Acceleration vs. proper time for the LDR solution (2.2), (2.3), showing pre-acceleration, predeceleration, and runaway behavior for $\tau < 0$. Here $a(\tau) \equiv \sqrt{a^2(\tau)}$.

hence are unobservable owing to masking by quantum field theory effects.[5] However, it is a consistent and tenable physical philosophy to demand that a theory be free of unphysical effects whatever the sizes of physical constants like $\hbar$. That is, a theoretical failing masked by another theory due to the "accidental" sizes of $\hbar$, c, e, m, etc., in this particular universe, should not be condoned.

Point (b) above means roughly the following. Expansions of the unknown $x^\mu(\tau; r_0)$ in a power series in the small parameter r_0 and solution of these perturbation theory equations is known to give a very good description of the experimentally observed motion,[6] for example, the radiation reaction and spiralling inward of a charge in a Coulomb field.[7] *However,* as is known, and as we shall explicitly show below, in general the solution of (1.1), (1.2) admits no such power series.[8] Thus there is no guarantee that the solution will agree even approximately with perturbation theory and hence with experimentally confirmed motions. Moreover, this is not the fault of the LDR data (1.2); it persists with any sort of unique specification. This will be made precise below in Section 3 and gives rise to an important open question.

[5] Rohrlich (1965), Section 6–7.

[6] Rohrlich (1965), p. 156. However, good data on charged particle orbits may be scanty.

[7] Rohrlich (1965), Section 6–15, pp. 184–185. It is not made very explicit that one is here solving by perturbation theory.

[8] For example, the runaway solution with $A \neq 0$.

2. *Illustration by an Exact Solution*

Specialize our finite duration force to a piecewise constant one[9]

$$\begin{aligned} A^{\mu}{}_{\nu} &= 0, \qquad t<0, \quad t_1<t \\ A^3{}_4(x) &= A^4{}_3(x) = f = \text{const.}, \qquad \text{other } A^{\mu}{}_{\nu}=0, \qquad 0<t<t_1 \end{aligned} \tag{2.1}$$

Then we have the exact solution[10]

$$v^{\mu}(\tau) = (0, 0, \sinh w(\tau), \cosh w(\tau)) \tag{2.2}$$

where

$$\begin{aligned} w(\tau) &= fr_0 e^{\tau/r_0}(1 - e^{-\tau_1/r_0}), \qquad \tau \leqslant 0 \\ w(\tau) &= f\tau + fr_0(1 - e^{-(\tau_1-\tau)/r_0}), \quad 0<\tau<\tau_1 \\ w(\tau) &= f\tau_1, \qquad \tau_1 \leqslant \tau \end{aligned} \tag{2.3}$$

corresponding to the LDR boundary conditions (1.2) with the initial data[3]

$$\mathbf{x}(\tau_1) \text{ arbitrary}, \qquad x^4(\tau_1) = t_1, \qquad v^{\mu}(-\infty) = (0001) \tag{2.4}$$

Here we see clearly the runaway behavior in $-\infty < \tau < 0$ because $a^{\mu}(0) \neq 0$, the preacceleration just before $\tau = 0$ and the predeceleration just before $\tau = \tau_1$ [cf. point (a) above]. As for point (b), this solution is *not* analytic at $r_0 = 0$; in fact it has an essential singularity there.

Since the exponential terms in (2.3) are appreciable only in the practically unobservable time intervals $|\tau| \leqslant r_0$ and $|\tau - \tau_1| \leqslant r_0 \approx 10^{-23}$ sec, we could define an "effective LDR solution" by simply deleting these terms. That is, in general

$$x^{\mu}_{\text{eff}}(\tau) \equiv x^{\mu}(\tau; e^{\tau/r_0} \to 0) \tag{2.5}$$

For the solution above this gives

$$\begin{aligned} w_{\text{eff}}(\tau) &= 0, \qquad \tau<0 \\ &= f\tau + fr_0, \qquad 0<\tau<\tau_1 \\ &= f\tau_1, \qquad \tau_1<\tau \end{aligned} \tag{2.6}$$

where v^{μ}_{eff} defined by (2.2) in terms of w_{eff}. *The meaning of the effective solution is that it describes the actual LDR solution away from certain critical proper times* τ_i (here $\tau = 0$ and τ_1), i.e., for all τ such that $|\tau - \tau_i| \gg r_0$, all i. One cannot mutilate the LDR solution further without destroying its

[9] This step function force can be rounded off to a continuous one, for which the existence and uniqueness theorems apply. Then the step function limit is taken in the solution. This has been done in (2.2) and (2.3). Note that $x^{\mu}(\tau)$, $v^{\mu}(\tau)$ are continuous (in fact the solution is C^{∞}).

[10] Rohrlich (1965), Section 6-13. We give only v^{μ} here and hereafter.

observable predictions. For example, one should not further remove the small but quite observable term fr_0, which acts over the whole macroscopic time interval $(0, \tau_1)$ and whose size, relative to the deleted term for that interval, is

$$e^{(\tau_1-\tau)/r_0} \ggg 1$$

for all of that interval except the last $\approx 10^{-23}$ sec.

We must now compare the LDR solutions, or their effective versions, with the predictions of perturbation theory. This will require a certain amount of care in making the latter precise.

3. *Perturbation Theory and an Open Question*

Let $l^{-1} \equiv$ size of $(q/m)\, F_{\text{ext}}$. Define the dimensionless position y^μ, proper time σ, and parameter of smallness ϵ by

$$y^\mu \equiv x^\mu/l, \qquad \sigma \equiv \tau/l, \qquad \epsilon \equiv r_0/l \tag{3.1}$$

Let $A^\mu{}_\nu \equiv (q/m)\, F_{\text{ext}}{}^\mu{}_\nu$ be piecewise C^∞ in x^λ in the intervals $t_i < t < t_{i+1}$, $i = 0, 1, \ldots$. Let Σ_i: $\sigma_i < \sigma < \sigma_{i+1}$ be the corresponding σ intervals, where $ly^4(\sigma_i) \equiv t_i$ and $y^\mu(\sigma)$ is the exact LDR solution of (1.1), (1.2). Then we define: *$x^\mu(\tau; r_0)$ is analytic in r_0 at $r_0 = 0$* iff $y^\mu(\sigma;\epsilon)$ has a power series in ϵ, with coefficient functions of σ, converging in some circle, for $\sigma \in$ each interval Σ_i. In particular

$$v(\tau; r_0) \equiv V(\sigma; \epsilon) = \sum_{p=0}^{\infty} \epsilon^p h_p{}^{(i)}(\sigma) \tag{3.2}$$

where $0 \leqslant |\epsilon| < E_i$ and $\sigma \in \Sigma_i$, for all i.

Perturbation Theory will mean the infinite set of second-order differential equations for each Σ_i obtained by substituting the power series into (1.1) and equating the coefficients of ϵ^p, $p = 0, 1, \ldots$, to zero[11]. As second-order equations, we need initial data[12] $\{x_0, v_0\}$ and the assumption *that $x(\tau)$, $v(\tau)$ are continuous* to make the solution unique. The perturbation theory solution describing the same problem as an exact LDR solution (1.1), (1.2) will have the same initial data $D_0 \equiv$ (1.2a).

In particular, for a finite duration, piecewise constant F_{ext} like (2.1), the perturbation theory equations are (writing $x = x_0 + x_1 + x_2 + \cdots$, where $x_p \propto \epsilon^p$)

$$\ddot{x}_0 = 0, \qquad \tau < 0, \qquad \tau_1 < \tau; \qquad \ddot{x}_0 = A_{op}\dot{x}_0, \qquad 0 < \tau < \tau_1 \tag{3.3$_0$}$$

[11] Fine points: (a) $lA^\mu{}_\nu(x_0) \equiv O(1)$ by definition, where $x_0 \equiv x_0{}^\mu(\tau)$ is the zero-order position. (b) We assume that the power series for y^μ can be differentiated term by term at least three times.

[12] Fine points: (a) The initial data are assumed analytic in r_0 at $r_0 = 0$ in the above sense. (b) If the initial data are given for $\sigma = \sigma_0 \in \Sigma_j$, say, then the assumed continuity of $x(\tau)$, $v(\tau)$ suffices to provide analytic initial data for the other intervals Σ_i, $i \neq j$, and hence to make the perturbation theory solution unique.

$$\ddot{x}_1 = 0, \qquad \tau < 0, \qquad \tau_1 < \tau; \qquad \ddot{x}_1 = r_0\Gamma_0 + A_{op}\dot{x}_1, \qquad 0 < \tau < \tau_1 \tag{3.3$_1$}$$

$$\vdots \qquad \text{etc.} \qquad \vdots$$

where A_{op} is the constant 4 x 4 matrix $(A^\mu{}_\nu)$ and Γ_0 means the $O(1)$ part of the Abraham vector Γ.

Now actually we know that, in general, solutions of the LDR theory are *not* analytic in r_0 at $r_0 = 0$ in this sense. In general, we expect singularities, in fact essential singularities, at $\epsilon = 0$. For example, the solution (2.2), (2.3), when phrased as a function of σ and ϵ, where we take $l^{-1} \equiv f$, has terms like $\epsilon e^{\sigma/\epsilon}$ in arguments of the hyperbolic functions. Hence no such power series (3.2) exist.

Nevertheless, as remarked already in Section 1, perturbation theory, e.g., the equations (3.3), considered say as a formal series of which one keeps in practice only the first few terms, seems to give a good description, without known exceptions, to the experimentally observed motion. (Note that l is typically a macroscopic length, e.g., 1 cm, while $r_0 \approx 10^{-13}$ cm, so that the expansion parameter $\epsilon \lll 1$.) *We take this as an indication that for the correct physical theory of the EM self-force, the solutions will either be analytic in r_0 at $r_0 = 0$ or (at least) the effective solutions, as defined in equation (2.5), will be analytic and moreover will agree with the perturbation theory solutions.* If both these conditions are met, we may call the theory "effectively analytic."

Hence we have the following open question about the present (LDR) theory of the EM self-force:

$$\text{open question: is the LDR theory effectively analytic?} \tag{3.4}$$

As far as we know, this question has never even been raised, much less answered.

In symbols, the open question reads

$$V_{\text{eff}}(\sigma;\epsilon) \stackrel{?}{=} V_{\text{PT}}(\sigma;\epsilon) \equiv \sum_{p=0}^{\infty} \epsilon^p h_p^{(i)}(\sigma), \qquad \sigma \in \Sigma_i, \text{all } i, \tag{3.5}$$

where $V(\sigma;\epsilon)$ is the LDR solution $v(\tau;r_0)$ expressed as a function of σ and ϵ, and $V_{\text{eff}}(\sigma;\epsilon)$ is obtained from it by setting all $e^{\sigma/\epsilon}$ equal to zero. The perturbation theory solution $V_{\text{PT}}(\sigma;\epsilon)$ is defined as the formal series of the perturbation equation solutions. Both solutions V and V_{PT} have the same initial data D_0.

We close by investigating whether in fact the particular solution (2.2), (2.3) is effectively analytic. v_{eff} is given by (2.6). We take $l^{-1} \equiv f$ so that $\sigma \equiv f\tau$, $\epsilon \equiv fr_0$.

Hence

$$
\begin{aligned}
V_{\text{eff}}(\sigma; \epsilon) &= (0001), \qquad \sigma < 0 \\
&= (0, 0, \sinh(\sigma + \epsilon), \cosh(\sigma + \epsilon)) \\
&= (0, 0, \sinh \sigma, \cosh \sigma) + \epsilon\,(0, 0, \cosh \sigma, \sinh \sigma) \\
&\qquad\qquad + O(\epsilon^2), \qquad 0 < \sigma < \sigma_1 \\
&= (0, 0, \sinh \sigma_1, \cosh \sigma_1), \qquad \sigma_1 < \sigma
\end{aligned}
\tag{3.6}
$$

It *is* analytic, as expected, and in the driven region II it has terms of $O(\epsilon^p)$ for all $p = 0, 1, \ldots$.

Look now at the perturbation theory equations (3.3) with the same external force (2.1) and the same initial data D_0, namely[3]

$$
\mathbf{x}_{\text{PT}}(\tau_1) \text{ arbitrary}, \qquad x^4_{\text{PT}}(\tau_1) = t_1, \qquad v^\mu_{\text{PT}}(-\infty) = (0001) \tag{3.7}
$$

In free region I the motion is uniform by (3.3), hence from (3.7)

$$
v_{\text{PT}}(\tau) = (0001), \qquad \tau < 0 \tag{3.8}
$$

In driven region II the x_0 motion is uniform acceleration along the z axis by (3.3_0). Therefore[13]

$$
v_0{}^\mu = \alpha_0{}^\mu e^{f\tau} + \beta_0{}^\mu e^{-f\tau}, \qquad 0 < \tau < \tau_1 \tag{3.9}
$$

where

$$
\alpha_0{}^\mu = (0, 0, \alpha_0, \alpha_0), \qquad \beta_0{}^\mu = (0, 0, -\beta_0, \beta_0) \tag{3.10}
$$

where α_0 and β_0 are arbitrary $O(1)$ constants, satisfies (3.3_0).

The normalization condition $v_{\text{PT}}^2 = -1$ with $v_{\text{PT}} = v_0 + v_1 + v_2 + \cdots$ becomes $v_0{}^2 + 2v_0 \cdot v_1 + \cdots = -1$, or

$$
v_0{}^2 = -1, \qquad v_0 \cdot v_1 = 0, \qquad \cdots \tag{3.11}
$$

Hence τ is x_0-proper time as well. The continuity of v_{PT} at $\tau = 0$: $v_{\text{PT}}(0+) = v_{\text{PT}}(0-) = (0001)$ in view of (3.8) yields

$$
v_0(0+) = (0001), \qquad v_1(0+) = 0, \ldots \tag{3.12}
$$

Now impose (3.11_0) and (3.12_0) to determine the constants α_0, β_0. This gives

$$
4\alpha_0\beta_0 = -1, \qquad \alpha_0 - \beta_0 = 0, \qquad \alpha_0 + \beta_0 = 1 \tag{3.13}
$$

or $\alpha_0 = \beta_0 = 1/2$.

But then $\dot{v}_0{}^2 = f^2$, $\ddot{v}_0 = f^2 v_0$, so that $\Gamma_0 \equiv (\ddot{v}_0 - \dot{v}_0{}^2 v_0) = 0$. Hence the v_1 equation in region II is

$$
\dot{v}_1 = A_{op} v_1, \qquad 0 < \tau < \tau_1 \tag{3.14}
$$

[13] Do not confuse the $O(1)$ velocity $v_0 \equiv v_0{}^\mu(\tau)$ with the initial data $\mathbf{v}_0 = \text{const}$.

The general solution of this is just like (3.9), (3.10) with $\alpha_0, \beta_0 \to \alpha_1, \beta_1 \equiv$ arbitrary $O(\epsilon)$ constants. Imposing (3.11_1) and (3.12_1) to determine the constants, we get

$$\alpha_0\beta_1 + \alpha_1\beta_0 = 0, \qquad \alpha_1 - \beta_1 = 0, \qquad \alpha_1 + \beta_1 = 0 \qquad (3.15)$$

or $\alpha_1 = \beta_1 = 0$, so $v_1(\tau) = 0$ in region II.

In free region III the motion is uniform by (3.3). Imposing continuity of $v_{\rm PT}$ at $\tau = \tau_1$: $v_{\rm PT}(\tau_1+) = v_{\rm PT}(\tau_1-) = v_0(\tau_1-) + v_1(\tau_1-) = (0, 0, \sinh f\tau_1, \cosh f\tau_1)$ to $O(\epsilon)$ in view of the solution in region II, we get $v_0(\tau) \doteq (0, 0, \sinh f\tau_1, \cosh f\tau_1)$, $v_1(\tau) = 0$, $\tau_1 < \tau$.

Therefore if we define $v'_{\rm PT} \equiv v_0 + v_1$, the complete PT solution up to $O(\epsilon)$[14] is then

$$\begin{aligned} V'_{\rm PT}(\sigma;\epsilon) \equiv v'_{\rm PT}(\tau; r_0) &= (0001), && \sigma \leqslant 0 \\ &= (0, 0, \sinh\sigma, \cosh\sigma), && 0 < \sigma < \sigma_1 \\ &= (0, 0, \sinh\sigma_1, \cosh\sigma_1), && \sigma_1 \leqslant \sigma \end{aligned} \qquad (3.16)$$

This is to be compared with (3.6), keeping terms up to $O(\epsilon)$. We see $V_{\rm PT}(\sigma;\epsilon)$ has no term of $O(\epsilon)$ in its power series, while $V_{\rm eff}(\sigma;\epsilon)$ does, in the interval $0 < \sigma < \sigma_1$. The upshot is that the solution (2.2), (2.3) of the LDR theory is *not* effectively analytic.

It would be interesting to investigate more complicated solutions than this one (piecewise constant electric field, rectilinear motion), such as cyclotron or Coulomb field motion, to see whether the violation of effective analyticity is worse. Note that in the above case at least the zero-order motions agree, and the motions could be made to agree by a simple (but unwarranted!) shift of the zero of proper time in the driven region $0 < \sigma < \sigma_1$, cf. (3.6). That is, both motions are uniformly accelerated, so they would be hard to distinguish experimentally. Though the solution is technically not effectively analytic by our definition, this is not very conclusive. Do there exist fields for which the two zero-order motions disagree or for which the $O(\epsilon)$ motions are of different character?

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[14] We can actually infer that (3.16) is the PT solution good to all orders, but we do not need this.